New reconstructions from cone Radon transform

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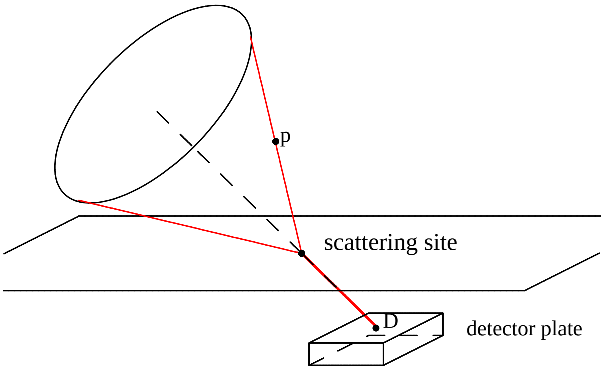
- Trajectories of single-scattered photons with fixed income and outcome energies in Compton camera form a cone of rotation:



Scheme of the Compton camera

- A spherical cone in an Euclidean space E^3 with apex at the origin can be written in the form

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$$g_C(y) = \cos \psi \int_{x \in C(\lambda)} f(y + x)\, w(x)\, \mathrm{d}x_2 \mathrm{d}x_3, \ y \in E^3$$

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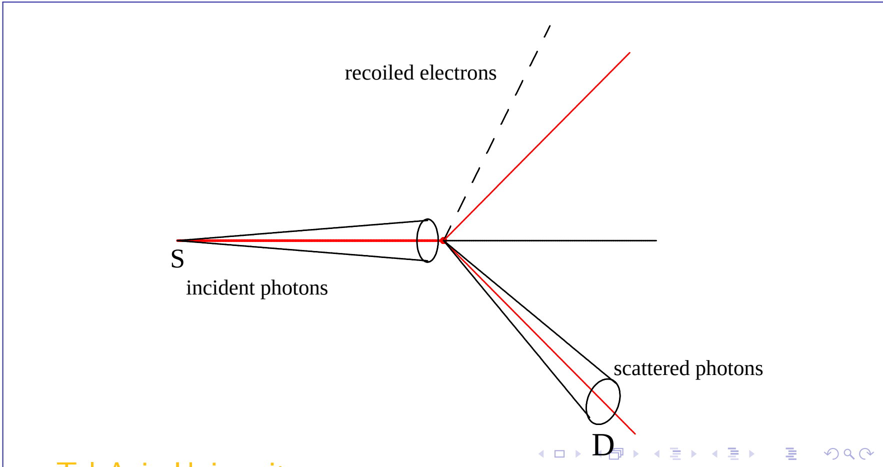
- Analytic inversion of the regular and singular monochrome (one opening) cone Radon transforms is in the focus of this talk.

Single-scattering tomography

- The realistic model (SPSF) for single-scattering optical tomography based on the photometric law of scattered radiation modeled by the singular cone transform.



Polychrome reconstructions

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- Haltmeier 2014, Terzioglu 2015, Moon 2016, Jung and Moon 2016 gave inversion formulae for arbitrary dimension n.
- Jung and Moon 2016 proposed the scheme for collecting non redunded data from a line of detectors and rotating axis.

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- Gouia-Zarrad and Ambartsoumian 2014 found the reconstruction formula for the regular cone transform in the half-space with free apex.

Cone transform with free apex

- Cone Radon integral equation can written in the convolution form

$$g = |x|^{-k} \delta_{-C} * f, \tag{1}$$

where

$$\begin{aligned} \delta_{-C}(\varphi) &= \int_C \varphi \mathrm{d}S = \cos^{-1}\psi \int\int \varphi(-\lambda s, x_2, x_3)\, \mathrm{d}x_2 \mathrm{d}x_3, \\ s &= \sqrt{x_2^2 + x_3^2}. \end{aligned}$$

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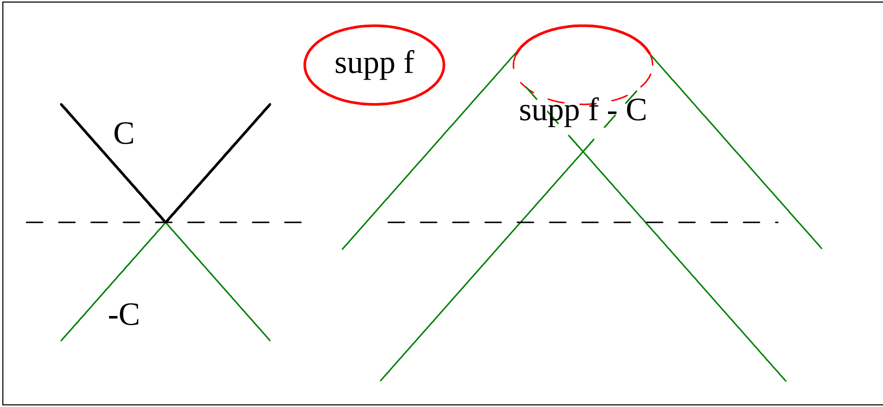
- We focus on the case $n = 3$ and use the notations

$$\Delta_0 = \delta_{-C},\ \Delta_1 = |x|^{-1}\,\delta_{-C}.$$

Support of the convolution

- For a function f on E^n vanishing for $x_1 > m$ for some m, the convolution $g = \Delta_k * f$ is well defined and $\mathrm{supp}\Delta_k * f \subset \mathrm{supp} f - C$.



Inversion of regular transforms

- **Case** $k = 0$. *The solution of*

$$\Delta_0 * f_0 = g_0,$$

can be found in the form

$$f_0(x) = \frac{1}{2\pi \cos^3 \psi} \square^2 \Delta_1 * \Theta_1 * g_0 \qquad (2)$$

$$= \frac{1}{2\pi \cos^3 \psi} \square^2 \int_{t \in C} \left(\int_{x_1}^{\infty} g_0(y - t_1, x_2 - t_2, x_3 - t_3) \, \mathrm{d}y \right) \frac{\mathrm{d}S}{|t|}$$

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and

- $$\Box = \frac{\partial^2}{\partial x_1^2} - \lambda^2\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right).$$

- **Case** $k = 1$. *The solution* of

$$\Delta_1 * f_1 = g_1 \tag{3}$$

reads

$$f_1(x) = \frac{1}{2\pi \cos^3 \psi} \square^2 \Delta_0 * \Theta_1 * g_1 \tag{4}$$

$$= \frac{1}{2\pi \cos^3 \psi} \square^2 \int_{t \in C} \int_{x_1}^{\infty} g_1 (y - t_1, x_2 - t_2, x_3 - t_3) \, \mathrm{d}y \mathrm{d}S.$$

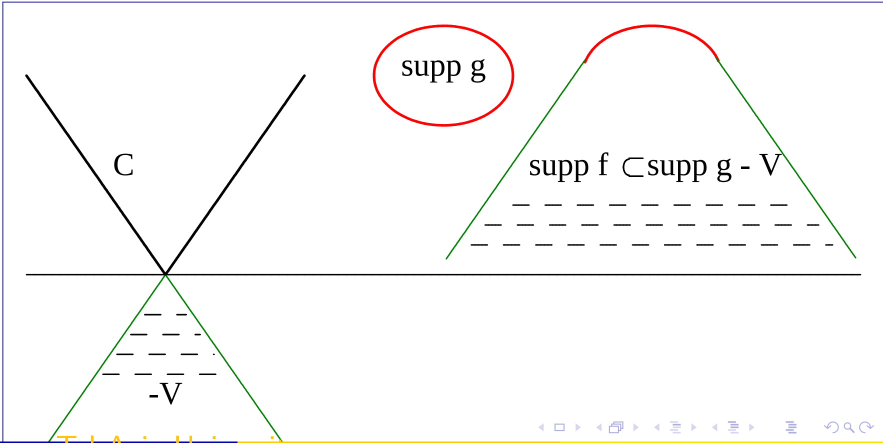
Conclusion: Inversion of any of two regular cone transform is given by the another cone transform followed (or preceded) by the 4 order differential operator and additional integration from x_1 to ∞ in the vertical variable. No Fourier transform etc. is necessary.

Support of the solution

- **Corollary** *For any function f with support in E_m for some m, we have*

$$\text{supp} f \subset \text{supp}\Delta_k * f - V, \; k = 0, 1$$

where V is the convex hull of C.



Proofs

- Distributions Δ_0 and Δ_1 are homogeneous of order 2 and 1. Fourier transforms are equal to (V.P. 2016, P.140)

$$\hat{\Delta}_0(p) = -\frac{1}{2\pi\cos^2\psi}\,|p_1|\left(p_1^2 - \lambda^2\left(p_2^2 + p_3^2\right)\right)^{-3/2},$$

$$\hat{\Delta}_1(p) = -\frac{2i}{\cos\psi}\,\mathrm{sgn} p_1\left(p_1^2 - \lambda^2\left(p_2^2 + p_3^2\right)\right)^{-1/2}$$

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- Both have analytical continuation at $H_+ = \left\{p \in \mathbb{C}^3 : \operatorname{Im} p_1 \geq 0\right\}$.

- The above calculations results

$$2\pi i \cos^3 \psi \left(p_1 + i0\right)^{-1} \left(p_1^2 - \lambda^2 \left(p_2^2 + p_3^2\right)\right)^2 \hat{\Delta}_0 \left(p\right) \hat{\Delta}_1 \left(p\right) = 1$$

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- Calculating the inverse Fourier transform we obtain

$$F^{-1} \left(p_1^2 - \lambda^2 \left(p_2^2 + p_3^2\right)\right) = -\frac{1}{4\pi^2} \Box\delta_0,$$

and

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where $\Theta_1 = \theta\left(x_1\right) \delta_0 \left(x_2, x_3\right)$, $\theta\left(t\right) = 1$ for $t < 0$ and $\theta\left(t\right) = 0$ for $t > 0$.

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- Finally

$$\cos^3\psi\Box^2\delta_0*\Theta_1*\Delta_1*\Delta_0=\delta_0, \tag{5}$$

where the convolutions of distributions Θ_1, Δ_1 and $\Box^2\delta_0$ are well defined and commute.

- Applying (5) to f_0 gives

$$f_0 = \cos^3\psi\Box^2 * \Delta_1 * \Theta_1 * \Delta_0 * f_0 = \cos^3\psi\Box^2 * \Delta_1 * \Theta_1 * g_0$$

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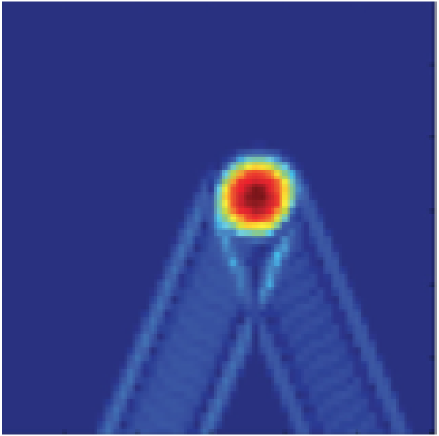
- **Remark 1.** Constant attenuation can be included in this method.

- **Remark 2.** Solution of (1) could be done in form $\hat{g}(p)/\hat{\Delta}_k(p)$ in the frequency domain. Iimplementation of this method supposes cutting out the "plumes" of g which causes the artifacts in the reconstruction as in the following picture



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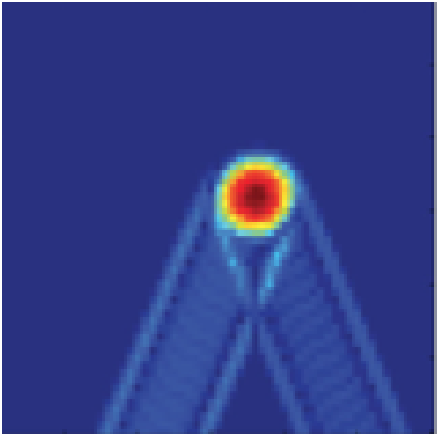

- which is due to the courtesy of Gouia-Zarrad, Ambartsoumian 2014.

Inversion of the singular cone transform

- Fix $\lambda > 0$ and consider the singular integral transform

$$G(q,\theta) = \int_{C_\lambda(\theta)} f(q+x) \frac{\mathrm{d}S}{|x|^2}, \ \theta \in \mathrm{S}^2, \ q \in E^3, \tag{6}$$

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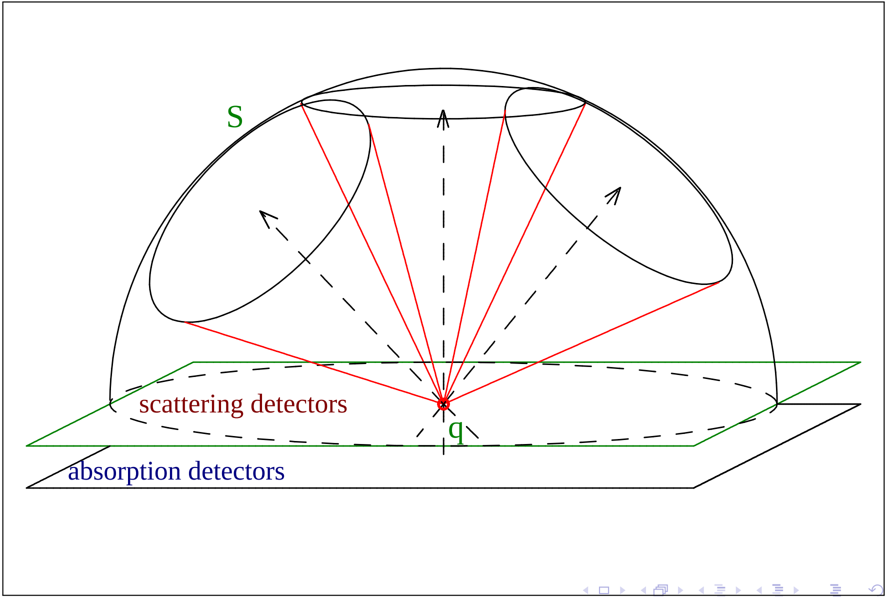
- (i) *any plane H which meets $\mathrm{supp} f$ has a common point with Q,*
- (ii) *for any point $q \in Q$, there exists a unit vector $\theta(q)$ such that $\mathrm{supp} f \subset q + C_\lambda(\theta(q))$.*

Compton cones with swinging axis



Proof

- **Step 1.** The singular ray transform

$$Xf(q,\xi)=\int_0^\infty f(q+r\xi)\frac{\mathrm{d}r}{r},\ \xi\in\mathrm{S}^2, q\in Q \tag{7}$$

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$$G\left(q,\theta\right)=\int_{\mathrm{S}_\lambda(\theta)}\int_0^\infty f\left(q+\xi\left(\sigma\right)r\right)\frac{\mathrm{d}r}{r}\mathrm{d}\sigma=\int_{\mathrm{S}_\lambda(\theta)}Xf\left(q,\xi\left(\sigma\right)\right)\mathrm{d}\sigma,$$

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  where $\xi(\sigma)$ runs over the circle $\mathrm{S}_\lambda(\theta) = C_\lambda(\theta) \cap \mathrm{S}^2$.

- Circles $\mathrm{S}_\lambda(\theta)$ have the same radius $r = \lambda\left(1 + \lambda^2\right)^{-1/2}$.

Proof

- **Step 1.** The singular ray transform

$$Xf(q,\xi) = \int_0^\infty f(q + r\xi)\frac{\mathrm{d}r}{r}, \ \xi \in \mathrm{S}^2, q \in Q \tag{7}$$

  is wel defined since f vanishes on Q since of (ii).

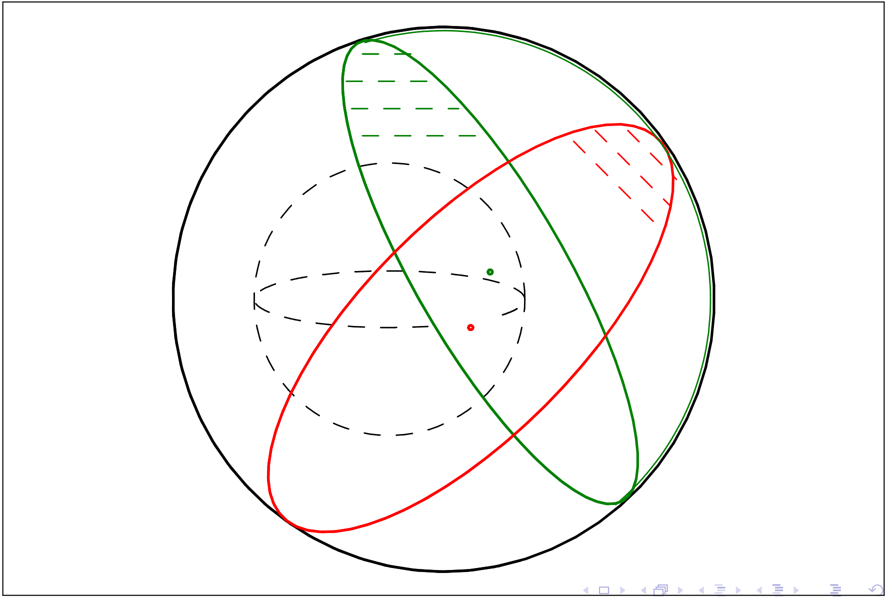
- By Fubini's theorem

$$G(q,\theta) = \int_{\mathrm{S}_\lambda(\theta)} \int_0^\infty f(q + \xi(\sigma) r)\frac{\mathrm{d}r}{r}\mathrm{d}\sigma = \int_{\mathrm{S}_\lambda(\theta)} Xf(q,\xi(\sigma))\,\mathrm{d}\sigma,$$

  where $\xi(\sigma)$ runs over the circle $\mathrm{S}_\lambda(\theta) = C_\lambda(\theta) \cap \mathrm{S}^2$.

- Circles $\mathrm{S}_\lambda(\theta)$ have the same radius $r = \lambda\left(1+\lambda^2\right)^{-1/2}$.
- The planes containing these circles are tangent to the central ball B of radius $\rho = \left(1+\lambda^2\right)^{-1/2}$.

Step 2: Nongeodesic Funk transform



- **Theorem** *For any* ρ, $0 \leq \rho < 1$, $\alpha \in E$, $|\alpha| \leq 1$, *an arbitrary function* $g \in C^2\left(\mathrm{S}^2\right)$ *can be reconstructed from data of integrals*

$$\Gamma(\theta) = \int_{\xi \in \mathrm{S}^2, \langle \xi - \alpha, \theta \rangle = \rho} g(\xi)\, \mathrm{d}\sigma, \ \ \theta \in \mathrm{S}^2 \tag{8}$$

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$$g(\xi) = -\frac{|\xi - \alpha|^2}{2\pi^2 \left(|\xi - \alpha|^2 - \rho^2\right)^{1/2}} \int_{\mathrm{S}^2} \frac{\Gamma(\theta)}{\left(\langle \xi - \alpha, \theta \rangle - \rho\right)^2} \mathrm{d}S \tag{9}$$

provided there exists a vector $\theta_0 \in S^2$ *such that* $\mathrm{supp} g \subset \left\{\xi \in \mathrm{S}^2 : \langle \xi - \alpha, \theta_0 \rangle \geq \rho\right\}$.

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- The singular integral is regularized as follows

$$\int_{\mathrm{S}^2} \frac{\Gamma(\theta)}{(\langle \xi - \alpha, \theta \rangle - \rho)^2} \mathrm{d}S = -\Delta(\theta) \int_{\mathrm{S}^2} \Gamma(\theta) \log(\langle \xi - \alpha, \theta \rangle - \rho)\, \mathrm{d}S.$$

References

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- The reconstruction from spherical integrals (8) on S^n was stated in V.P. 2016 for arbitrary n, $0 \leq \rho < 1$, $|\alpha| \leq 1$.

- **Step 3.** By (i) formula (9) can be applied to $\Gamma(q,\theta)$ for $\alpha = 0$, $\rho = \left(1+\lambda^2\right)^{-1/2}$ which provides the reconstruction of $g(q,\xi) = Xf(q,\xi)$ for any $q \in Q$ and all ξ.

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$$\left\langle \omega, \nabla_\xi \right\rangle^2 f(q+r\xi) = r^2 \left\langle \omega, \nabla_q \right\rangle^2 f(q+r\xi)$$

which yields (by Grangeat's method) for any p,

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- where the left hand side can be calculated from Xf.

- **Step 4** By (ii) we can use the Lorentz-Radon formula for any $x \in \mathrm{supp} f$,

$$\begin{aligned} f(x) &= -\frac{1}{8\pi^2}\int_{\omega\in\mathrm{S}^2}\frac{\partial^2}{\partial p^2}\int_{\langle\omega,q-x\rangle=0} f(q)\,\mathrm{d}q\mathrm{d}\Omega \\ &= -\frac{1}{8\pi^2}\int_{\omega\in\mathrm{S}^2}\int_{\langle\omega,\xi\rangle=0}\langle\omega,\nabla_\xi\rangle^2 Xf(q(\omega),\xi)\,\mathrm{d}\varphi\mathrm{d}\Omega \end{aligned}$$

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- This completes the reconstruction of f.

Other reconstructions from the singular cone beam transform

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- **Theorem** *For any function* $f \in C_0^2\left(E^3\right)$ *and any* $x \in \operatorname{supp} f \backslash \Gamma$ *such that any plane* P *through* x *meets* Γ, *the equation holds*

$$\begin{aligned} f(x) &= -\frac{1}{32\pi^4} \int_{y\in\Gamma} \int_{\langle y-x,\xi\rangle=0} \partial_s^2 \frac{\varepsilon(y,\xi)}{|y-x|} \mathrm{d}s \\ &\quad \times \int_{\langle \xi, v\rangle = 0} \langle \xi, \nabla_v \rangle^2 \partial_s g(y, v)\, \mathrm{d}\theta \mathrm{d}\varphi. \end{aligned}$$

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